A perturbed differential resultant based implicitization algorithm for linear DPPEs*

Sonia L. Rueda
Dpto de Matemática Aplicada, E.T.S. Arquitectura.
Universidad Politécnica de Madrid.
Avda. Juan de Herrera 4, 28040-Madrid, Spain.
sonialuisa.rueda@upm.es

Abstract

Let $\mathbb{K}$ be an ordinary differential field with derivation ∂. Let $\mathcal{P}$ be a system of n linear differential polynomial parametric equations in $n-1$ differential parameters with implicit ideal ID. Given a nonzero linear differential polynomial A in ID we give necessary and sufficient conditions on A for $\mathcal{P}$ to be $n-1$ dimensional. We prove the existence of a linear perturbation $\mathcal{P}_\phi$ of $\mathcal{P}$ so that the linear complete differential resultant $\partial\mathrm{CRes}_\phi$ associated to $\mathcal{P}_\phi$ is nonzero. A nonzero linear differential polynomial in ID is obtained from the lowest degree term of $\partial\mathrm{CRes}_\phi$ and used to provide an implicitization algorithm for $\mathcal{P}$.

1 Introduction

The use of algebraic elimination techniques such as Groebner basis and multivariate resultants to obtain the implicit equation of a unirational algebraic variety is well known (see for instance [4], [5]). The development of similar techniques in the differential case is an active field of research. In [7] characteristic set methods were used to solve the differential implicitization problem for differential rational parametric equations and alternative methods are emerging to treat special cases. In [13], linear complete differential resultants were used to compute the implicit equation of a set of linear differential polynomial parametric equations (linear DPPEs). As in the algebraic case differential resultants often vanish under specialization and we are left with no candidate

*Supported by the Spanish "Ministerio de Ciencia e Innovación" under the Project MTM2008-04699-C03-01

to be the implicit equation. This reason prevented us from giving an algorithm for differential implicitization in [13]. Motivated by Canny's method in [2] and its generalizations in [6] and [10] in the present work we consider a linear perturbation of a given system of linear DPPEs and use linear complete differential resultants to give a candidate to be the implicit equation of the system.

Given a system $\mathcal{P}(X,U)$ of n linear ordinary differential polynomial parametric equations $x_1 = P_1(U), \ldots, x_n = P_n(U)$ in $n-1$ differential parameters $u_1, \ldots, u_{n-1}$ (we give a precise statement of the problem in Section 2) we give an algorithm to decide if the dimension of the implicit ideal ID of $\mathcal{P}$ is $n-1$ and in the affirmative case provide the implicit equation of $\mathcal{P}$.

The linear complete differential resultant $\partial\mathrm{CRes}(x_1 - P_1(U), \ldots, x_n - P_n(U))$ is the algebraic resultant of Macaulay of a set of differential polynomials with L elements. It was defined in [13] as a generalization of Carra'Ferro's differential resultant [3] (in the linear case) in order to adjust the number L of differential polynomials to the order of derivation of the variables $u_1, \ldots, u_n$ in $F_i = x_i - P_i(U)$.

In this paper, we provide a perturbation $\mathcal{P}_\phi(X,U)$ of $\mathcal{P}(X,U)$ so that the linear differential polynomials $F_1 - p\,\phi_1(U), \ldots, F_n - p\,\phi_n(U)$ have nonzero linear complete differential resultant $\partial\mathrm{CRes}_\phi(p)$ which is a polynomial depending on p. It will be shown that the coefficient of the lowest degree term of $\partial\mathrm{CRes}_\phi(p)$ is a nonzero linear differential polynomial that belongs to the implicit ideal ID of $\mathcal{P}(X,U)$. In fact, if $\partial\mathrm{CRes}_\phi(p)$ has a nonzero independent term it equals $\partial\mathrm{CRes}(F_1, \ldots, F_n)$ and as proved in [13] it gives the implicit equation of $\mathcal{P}(X,U)$.

The main result of this paper generalizes the result previously mentioned from [13]. Given a nonzero linear differential polynomial A in ID necessary and sufficient conditions on A are provided so that $A(X) = 0$ is the implicit equation of $\mathcal{P}(X,U)$. The higher order terms in the equations of $\mathcal{P}(X,U)$ and the rank of the coefficient matrix of the set of L polynomials used to construct the differential resultant $\partial\mathrm{CRes}(F_1, \ldots, F_n)$ play a significant role in this theory. The fact that we are dealing with linear differential polynomials will be also relevant, allowing us to treat then using differential operators.

The paper is organized as follows. In Section 2 we introduce the main notions and notation. Next we review the definition of the linear complete differential (homogeneous) resultant in Section 3. The main definitions regarding linear differential polynomials in ID are given in Section 4. The next section contains the main result of the paper, namely a characterization of ID in the $n-1$ dimensional case is provided in Section 5. In Section 6 we give a perturbation $\mathcal{P}_\phi(X,U)$ of $\mathcal{P}(X,U)$ with nonzero differential resultant and use it to

obtain a nonzero linear differential polynomial in ID candidate to provide the implicit equation. Finally in Section 7 we give the implicitization algorithm and examples.

2 Basic notions and notation

In this section, we introduce the basic notions related to the problem we deal with (as in [12] and [13]), as well as notation and terminology used throughout the paper. For further concepts and results on differential algebra we refer to [8] and [9].

Let $\mathbb{K}$ be an ordinary differential field with derivation ∂, (e.g. $\mathbb{Q}(t)$, $\partial = \frac{\partial}{\partial t}$). Let $X = \{x_1, \ldots, x_n\}$ and $U = \{u_1, \ldots, u_{n-1}\}$ be sets of differential indeterminates over $\mathbb{K}$. Let $\mathbb{N}_0 = \{0, 1, 2, \ldots, n, \ldots\}$. For $k \in \mathbb{N}_0$ we denote by x_{ik} the k-th derivative of x_i, for x_{i0} we simply write x_i. Given a set Y of differential indeterminates over $\mathbb{K}$ we denote by $\{Y\}$ the set of derivatives of the elements of Y, $\{Y\} = \{\partial^k y \mid y \in Y,\ k \in \mathbb{N}_0\}$, and by $\mathbb{K}\{X\}$ the ring of differential polynomials in the differential indeterminates $x_1, \ldots, x_n$, that is

$$\mathbb{K}\{X\} = \mathbb{K}[x_{ik} \mid i = 1, \ldots, n,\ k \in \mathbb{N}_0].$$

Analogously for $\mathbb{K}\{U\}$.

As defined in [13] we consider the system of linear DPPEs

$$\mathcal{P}(X, U) = \begin{cases} x_1 & = & P_1(U) \\ & \vdots & \\ x_n & = & P_n(U). \end{cases} \tag{1}$$

where $P_1, \ldots, P_n \in \mathbb{K}\{U\}$ with degree at most 1 and not all $P_i \in \mathbb{K}$, $i = 1, \ldots, n$. There exists $a_i \in \mathbb{K}$ and an homogeneous differential polynomial $H_i \in \mathbb{K}\{U\}$ such that

$$F_i(X, U) = x_i - P_i(U) = x_i - a_i + H_i(U).$$

Given $P \in \mathbb{K}\{X \cup U\}$ and $y \in X \cup U$, we denote by $\mathrm{ord}(P, y)$ the order of P in the variable y. If P does not have a term in y then we define $\mathrm{ord}(P, y) = -1$. To ensure that the number of parameters is $n - 1$, we assume that for each $j \in \{1, \ldots, n-1\}$ there exists $i \in \{1, \ldots, n\}$ such that $\mathrm{ord}(F_i, u_j) \geq 0$.

The implicit ideal of the system (1) is the differential prime ideal

$$\mathrm{ID} = \{f \in \mathbb{K}\{X\} \mid f(P_1(U), \ldots, P_n(U)) = 0\}.$$

Given a characteristic set $\mathcal{C}$ of ID then $n- \mid \mathcal{C} \mid$ is the (differential) dimension of ID, by abuse of notation, we will also speak about the dimension of a DPPEs system meaning the dimension of its implicit ideal.

If $\dim(\mathrm{ID}) = n-1$, then $\mathcal{C} = \{A(X)\}$ for some irreducible differential polynomial $A \in \mathbb{K}\{X\}$. The polynomial A is called a characteristic polynomial of ID. The implicit equation of a $(n-1)$-dimensional system of DPPEs, in n differential indeterminates $X = \{x_1, \ldots, x_n\}$, is defined as the equation $A(X) = 0$, where A is any characteristic polynomial of the implicit ideal ID of the system.

Let $\mathbb{K}[\partial]$ be the ring of differential operators with coefficients in $\mathbb{K}$. If $\mathbb{K}$ is not a field of constants with respect to ∂, then $\mathbb{K}[\partial]$ is not commutative but $\partial k - k\partial = \partial(k)$ for all $k \in \mathbb{K}$. The ring $\mathbb{K}[\partial]$ of differential operators with coefficients in $\mathbb{K}$ is left euclidean (and also right euclidean). Given $\mathcal{L}, \mathcal{L}' \in \mathbb{K}[\partial]$, by applying the left division algorithm we obtain $q, r \in \mathbb{K}[\partial]$, the left quotient and the left reminder of $\mathcal{L}$ and $\mathcal{L}'$ respectively, such that $\mathcal{L} = \mathcal{L}'q + r$ where $\deg(r) < \deg(\mathcal{L}')$.

3 Linear complete differential resultants

We review next the results on linear complete differential resultants from [13] that will be used in this paper.

Let $\mathbb{D}$ be a differential integral domain, and let $f_i \in \mathbb{D}\{U\}$ be a linear ordinary differential polynomial of order o_i, $i = 1, \ldots, n$. For each $j \in \{1, \ldots, n-1\}$ we define the positive integers

$$\gamma_j(f_1, \ldots, f_n) := \min\{o_i - \mathrm{ord}(f_i, u_j) \mid i \in \{1, \ldots, n\}\},$$
$$\gamma(f_1, \ldots, f_n) := \sum_{j=1}^{n-1} \gamma_j(f_1, \ldots, f_n).$$

Let $N = \sum_{i=1}^{n} o_i$ then the completeness index $\gamma(f_1, \ldots, f_n)$ verifies $\gamma(f_1, \ldots, f_n) \leq N - o_i$, for all $i \in \{1, \ldots, n\}$.

The linear complete differential resultant $\partial\mathrm{CRes}(f_1, \ldots, f_n)$ is the Macaulay's algebraic resultant of the differential polynomial set

$$\mathrm{PS}(f_1, \ldots, f_n) :=$$
$$\{\partial^{N-o_i-\gamma} f_i, \ldots, \partial f_i, f_i \mid i = 1, \ldots, n,\ \ \gamma = \gamma(f_1, \ldots, f_n)\}.$$

The set $\mathrm{PS}(f_1, \ldots, f_n)$ contains $L = \sum_{i=1}^{n}(N - o_i - \gamma + 1)$ polynomials in the following set $\mathcal{V}$ of $L-1$ differential variables

$$\mathcal{V} = \{u_j, u_{j1} \ldots, u_{jN-\gamma_j-\gamma} \mid \gamma_j = \gamma_j(f_1, \ldots, f_n), j = 1, \ldots, n-1\}.$$

Let $h_i \in \mathbb{D}\{U\}$ be a linear ordinary differential homogeneous polynomial of order o_i, $i = 1, \ldots, n$ with $N = \sum_{i=1}^{n} o_i \geq 1$. We define the differential

polynomial set

$$\mathrm{PS}^h(h_1,\ldots,h_n) := \{\partial^{N-o_i-\gamma-1}h_i,\ldots,\partial h_i, h_i \mid$$
$$i\in\{1,\ldots,n\},\ \ N-o_i-\gamma-1\geq 0,\ \ \gamma=\gamma(h_1,\ldots,h_n)\}.$$

Observe that $N \geq 1$ implies $\mathrm{PS}^h(h_1,\ldots,h_n) \neq \emptyset$. The linear complete differential homogenous resultant $\partial\mathrm{CRes}^h(h_1,\ldots,h_n)$ is the Macaulay's algebraic resultant of the set $\mathrm{PS}^h(h_1,\ldots,h_n)$. The set $\mathrm{PS}^h(h_1,\ldots,h_n)$ contains $L^h = \sum_{i=1}^{n}(N-o_i-\gamma)$ polynomials in the set $\mathcal{V}^h$ of L^h differential variables

$$\mathcal{V}^h = \{u_j, u_{j1}\ldots,u_{jN-\gamma_j-\gamma-1} \mid \gamma_j=\gamma_j(h_1,\ldots,h_n), j=1,\ldots,n-1\}.$$

We review next the matrices that will allow the use of determinants to compute $\partial\mathrm{CRes}(f_1,\ldots,f_n)$ and $\partial\mathrm{CRes}^h(h_1,\ldots,h_n)$. The order $u_1 < \cdots < u_{n-1}$ induces an orderly ranking on U (i.e. and order on $\{U\}$) as follows (see [8], page 75): $u < \partial u$, $u < u^\star \Rightarrow \partial u < \partial u^\star$ and $k < k^\star \Rightarrow \partial^k u < \partial^{k^\star} u^\star$, for all $u, u^\star \in U$, $k, k^\star \in \mathbb{N}_0$. We set $1 < u_1$.

For $i = 1,\ldots,n$, $\gamma = \gamma(f_1,\ldots,f_n)$ and $k = 0,\ldots,N-o_i-\gamma$ define de positive integer $l(i,k) = (i-1)(N-\gamma) - \sum_{h=1}^{i-1} o_i + i + k$ in $\{1,\ldots,L\}$. The complete differential resultant matrix $M(L)$ is the $L\times L$ matrix containing the coefficients of $\partial^{N-o_i-\gamma-k}f_i$ as a polynomial in $\mathbb{D}[\mathcal{V}]$ in the $l(i,k)$-th row, where the coefficients are written in decreasing order with respect to the orderly ranking on U. In this situation:

$$\partial\mathrm{CRes}(f_1,\ldots,f_n) = \det(M(L)).$$

If $N \geq 1$, for $\gamma=\gamma(h_1,\ldots,h_n)$, $i\in\{1,\ldots,n\}$ such that $N-o_i-\gamma-1\geq 0$ and $k = 0,\ldots,N-o_i-\gamma-1$ define the positive integer $l^h(i,k) = (i-1)(N-\gamma-1) - \sum_{h=1}^{i-1} o_i + i + k$ in $\{1,\ldots,L^h\}$. The complete differential homogeneous resultant matrix $M(L^h)$ is the $L^h \times L^h$ matrix containing the coefficients of $\partial^{N-o_i-\gamma-k-1}h_i$ as a polynomial in $\mathbb{D}[\mathcal{V}^h]$ in the $l^h(i,k)$-th row, where the coefficients are written in decreasing order with respect to the orderly ranking on U. In this situation:

$$\partial\mathrm{CRes}^h(h_1,\ldots,h_n) = \det(M(L^h)).$$

Throughout the remaining parts of the paper we will say differential (homogeneous) resultant always meaning linear complete differential (homogeneous) resultant.

3.1 Linear complete differential resultants from linear DPPEs

We highlight in this section some facts on differential resultants of the differential polynomials F_i and H_i obtained from a system of linear DPPEs as in Section 2.

Let $\gamma = \gamma(F_1, \ldots, F_n) = \gamma(H_1, \ldots, H_n)$ and $\mathbb{D} = \mathbb{K}\{X\}$. Then $\partial\mathrm{CRes}(F_1, \ldots, F_n)$ and $\partial\mathrm{CRes}^h(H_1, \ldots, H_n)$ are closely related as shown in [13], Section 5. Since $F_i(X, U) = x_i - a_i + H_i(U)$, if $N \geq 1$ the matrix $M(L^h)$ is a submatrix of $M(L)$ obtained removing n specific rows and columns. This fact together with the identities below allowed to prove that (when $N \geq 1$)

$$\partial\mathrm{CRes}(F_1, \ldots, F_n) = 0 \Leftrightarrow \partial\mathrm{CRes}^h(H_1, \ldots, H_n) = 0.$$

The next matrices will play an important role in the remaining parts of the paper.

- Let S be the $n \times (n-1)$ matrix whose entry (i, j) is the coefficient of $u_{n-j\, o_i - \gamma_j}$ in F_i, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n-1\}$. We call S the leading matrix of $\mathcal{P}(X, U)$. For $i \in \{1, \ldots n\}$ let S_i be the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row of S.
- Let M_{L-1} be the $L \times (L-1)$ principal submatrix of $M(L)$. We call M_{L-1} the principal matrix of $\mathcal{P}(X, U)$.

Let $\mathcal{X} = \{x_i, x_{i1}, \ldots, x_{i\, N-o_i-\gamma} \mid i = 1, \ldots, n\}$. Given $x \in \mathcal{X}$, say $x = x_{ik}$ with $k \in \{0, 1, \ldots, N - o_i - \gamma\}$, let us call M_x the submatrix of M_{L-1} obtained by removing the row corresponding to the coefficients of $\partial^k F_i = x_{ik} + \partial^k(H_i(U) - a_i)$. Then, developing the determinant of $M(L)$ by the last column we obtain

$$\partial\mathrm{CRes}(F_1, \ldots, F_n) = \sum_{i=1}^{n} \sum_{k=0}^{N-o_i-\gamma} b_{ik} \det(M_{x_{ik}})(x_{ik} - \partial^k a_i), \tag{2}$$

with $b_{ik} = \pm 1$ according to the row index of $x_{ik} - \partial^k a_i$ in the matrix $M(L)$. Also, for every $i \in \{1, \ldots, n\}$, there exists $a \in \mathbb{N}$ such that

$$\det(M_{x_{iN-o_i-\gamma}}) = (-1)^a \partial\mathrm{CRes}^h(H_1, \ldots, H_n) \det(S_i). \tag{3}$$

4 The implicit ideal ID

Let $\mathcal{P}(X, U)$, F_i, H_i be as in Section 2. Let $\mathrm{PS} = \mathrm{PS}(F_1, \ldots, F_n)$ and let ID be the implicit ideal of $\mathcal{P}(X, U)$. In this section, we review the computation of ID in terms of characteristic sets (see [7] and [13]) and give some definitions related with linear differential polynomials in ID that will be important in the remaining parts of the paper.

Let $\mathcal{X}$ and $\mathcal{V}$ be as in Section 3.1 and observe that $\mathrm{PS} \subset \mathbb{K}[\mathcal{X}][\mathcal{V}]$. Let (PS) be the ideal generated by PS in $\mathbb{K}[\mathcal{X}][\mathcal{V}]$ and let [PS] be the differential ideal generated by PS in $\mathbb{K}\{X\}$.

The order $x_n < \cdots < x_1$ induces a ranking on X as follows (see [8], page 75): $x < \partial x$ and $x < x^\star \Rightarrow \partial^k x < \partial^{k^\star} x^\star$, for all $x, x^\star \in X$, $k, k^\star \in \mathbb{N}_0$.

- We call $\mathcal{R}$ the ranking on $X \cup U$ that eliminates X with respect to U, that is $\partial^k x > \partial^{k^\star} u$, for all $x \in X$, $u \in U$ and $k, k^\star \in \mathbb{N}_0$.
- We call $\mathcal{R}^\star$ the ranking on $X \cup U$ that eliminates U with respect to X, that is $\partial^k x < \partial^{k^\star} u$, for all $x \in X$, $u \in U$ and $k, k^\star \in \mathbb{N}_0$.

Note that, because of the particular structure of F_i, with respect to $\mathcal{R}$ then PS is a chain (see [9], page 3) of L differential polynomials, with L as in Section 3. Let $\mathcal{A}$ be a characteristic set of [PS] and $\mathcal{A}_0 = \mathcal{A} \cap \mathbb{K}\{X\}$. By [7] then

$$\mathrm{ID} = [\mathrm{PS}] \cap \mathbb{K}\{X\} = [\mathcal{A}_0].$$

To compute a characteristic set of [PS] we will use the reduced Groebner basis of (PS) with respect to lex monomial order induced by the ranking $\mathcal{R}^\star$. We call $\mathcal{G}$ the Groebner basis associated to the system $\mathcal{P}(X,U)$. For that purpose we apply the algorithm given in [1], Theorem 6, that we briefly include below for completion.

Given $P \in \mathbb{K}\{X \cup U\}$, the lead of P is the highest derivative present in P w.r.t. $\mathcal{R}^\star$, we denote it by $\mathrm{lead}(P)$. Given $P, Q \in \mathbb{K}\{X \cup U\}$ we denote by $\mathrm{prem}(P,Q)$ the pseudo-remainder of P with respect to Q, [9], page 7. Given a chain $\mathcal{A} = \{A_1, \ldots, A_t\}$ of elements of $\mathbb{K}\{X \cup U\}$ then $\mathrm{prem}(P, \mathcal{A}) = \mathrm{prem}(\mathrm{prem}(P, A_t), \{A_1, \ldots, A_{t-1}\})$ and $\mathrm{prem}(P, \emptyset) = P$.

Algorithm 4.1. Given *the set of differential polynomials* PS *the next algorithm* returns *a characteristic set* $\mathcal{A}$ *of* [PS].

1. *Compute the reduced Groebner basis* $\mathcal{G}$ *of* (PS) *with respect to lex monomial order induced by* $\mathcal{R}^\star$.
2. *Assume that the elements of* $\mathcal{G}$ *are arranged in increasing order* $B_0 < B_1 < \cdots < B_{L-1}$ *w.r.t.* $\mathcal{R}^\star$. *Let* $\mathcal{A}^0 = \{B_0\}$. *For* i *from* 1 *to* $L-1$ *do, if* $\mathrm{lead}(B_i) \neq \mathrm{lead}(B_{i-1})$ *then* $\mathcal{A}^i := \mathcal{A}^{i-1} \cup \{\mathrm{prem}(B_i, \mathcal{A}^{i-1})\}$. $\mathcal{A} := \mathcal{A}^{L-1}$.

We are dealing with a linear system of polynomials and computing a Groebner basis is equivalent to performing gaussian elimination. Some details on this computation were given in [13] and we include them below to be used further in this paper. Let M_{2L} be the $L \times (2L)$ matrix whose k-th row contains the coefficients, as a polynomial in $\mathbb{K}[\mathcal{X}][\mathcal{V}]$, of the $(L-k+1)$-th polynomial in PS and where the coefficients are written in decreasing order w.r.t. $\mathcal{R}^\star$.

$$M_{2L} = \begin{bmatrix} & 1 & & & & & & \partial^{N-o_1-\gamma}a_1 \\ & & \ddots & & & & & \vdots \\ & & & 1 & & & & a_1 \\ M_{L-1} & & & & \ddots & & & \vdots \\ & & & & & 1 & & \partial^{N-o_n-\gamma}a_n \\ & & & & & & \ddots & \vdots \\ & & & & & & 1 & a_n \end{bmatrix}.$$

The polynomials corresponding to the rows of the reduced echelon form E_{2L} of M_{2L} are the elements of the Groebner basis associated to $\mathcal{P}(X,U)$. Let $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. From [13], Lemma 20(1):

Lemma 4.2. *The Groebner basis associated to the system $\mathcal{P}(X,U)$ is a set of linear differential polynomials $\mathcal{G} = \{B_0, B_1, \ldots, B_{L-1}\}$ where $B_0 < B_1 < \cdots < B_{L-1}$ with respect to the ranking $\mathcal{R}^\star$ and $B_0 \in \mathcal{G}_0$. Hence* $(\mathrm{PS}) \cap \mathbb{K}\{X\} = (\mathcal{G}_0)$ *is nonzero.*

4.1 On linear differential polynomials in ID

The next definitions will play an important role throughout the paper. Given a nonzero linear differential polynomial B in [PS] there exist differential operators $\mathcal{F}_i \in \mathbb{K}[\partial]$, $i = 1, \ldots, n$ such that

$$B(X,U) = \sum_{i=1}^{n} \mathcal{F}_i(F_i(X,U)).$$

If B belongs to $\mathrm{ID} = [\mathrm{PS}] \cap \mathbb{K}\{X\}$ then

$$B = \sum_{i=1}^{n} \mathcal{F}_i(x_i - a_i) \text{ and } \sum_{i=1}^{n} \mathcal{F}_i(H_i(U)) = 0.$$

If B belongs to (PS) then $\deg(\mathcal{F}_i) \leq N - o_i - \gamma$, $i = 1, \ldots, n$.

Definition 4.3. *Given a nonzero linear differential polynomial B in (PS) with $B(X,U) = \sum_{i=1}^{n} \mathcal{F}_i(F_i(X,U))$, $\mathcal{F}_i \in \mathbb{K}[\partial]$.*

1. *We define the* co-order *of B in* (PS) *as the highest positive integer $\mathrm{c}(B)$ such that $\partial^{\mathrm{c}(B)} B \in$* (PS). *Observe that*

$$\mathrm{c}(B) = \min\{N - o_i - \gamma - \mathrm{ord}(B, x_i) \mid i = 1, \ldots, n\}.$$

2. *For $i \in \{1, \ldots, n\}$ let α_i be the coefficient of $\partial^{N-o_i-\gamma-\mathrm{c}(B)}$ in $\mathcal{F}_i$. We call $(\alpha_1, \ldots, \alpha_n)$ the* leading coefficients vector *of B in* (PS) *and we denote it by $\mathrm{l}(B)$.*

Let S be the leading matrix of the system $\mathcal{P}(X,U)$. Denote by S^T the transpose matrix of S.

Remark 4.4. *The i-th row of S equals* $\mathrm{l}(F_i(X,U))$, $i \in \{1,\ldots,n\}$. *Given a nonzero linear* $B \in (PS)$, *the i-th row of S also consists of the coefficients of* $u_{n-j\, N-\gamma_j-\gamma-\mathrm{c}(B)}$, $j \in \{1,\ldots,n-1\}$ *in* $\partial^{N-o_i-\gamma-\mathrm{c}(B)} F_i(X,U)$. *Therefore if* $\mathrm{ord}(B,u_j) < N-\gamma_j-\gamma-\mathrm{c}(B)$ *then* $\mathrm{l}(B)S = 0$, *that is* $\mathrm{l}(B) \in \mathrm{Ker}(S^T)$.

Definition 4.5. *Given a nonzero linear differential polynomial B in* ID *with* $B = \sum_{i=1}^n \mathcal{F}_i(x_i - a_i)$, $\mathcal{F}_i \in \mathbb{K}[\partial]$.

1. *We define the* ID-content *of B as the greatest common left divisor of* $\mathcal{F}_1,\ldots,\mathcal{F}_n$ *(we write* $\mathrm{gcld}(\mathcal{F}_1,\ldots,\mathcal{F}_n)$*). We denote it by* $\mathrm{IDcont}(B)$.

2. *There exist* $\mathcal{L}_i \in \mathbb{K}[\partial]$ *such that* $\mathcal{F}_i = \mathrm{IDcont}(B)\mathcal{L}_i$, $i = 1,\ldots,n$ *and* $\mathcal{L}_1,\ldots,\mathcal{L}_n$ *are coprime (we write* $(\mathcal{L}_1,\ldots,\mathcal{L}_n) = 1$*). We define the* ID-primitive part *of B as*

$$\mathrm{IDprim}(B)(X,U) = \sum_{i=1}^n \mathcal{L}_i(x_i - a_i).$$

3. *If* $\mathrm{IDcont}(B) \in \mathbb{K}$ *then we day that B is* ID-primitive .

If $A = \mathrm{IDprim}(B) = \sum_{i=1}^n \mathcal{L}_i(x_i - a_i)$ then $\mathrm{c}(A) \geq \deg(\mathrm{IDcont}(B))$ and $\deg(\mathcal{L}_i) \leq N - o_i - \gamma - \mathrm{c}(A)$, $i = 1,\ldots,n$.

Lemma 4.6. *Given a linear differential polynomial* $B \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$ *then* $\mathrm{IDprim}(B) \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$.

Proof. For $i = 1,\ldots n$ and $j = 1,\ldots n-1$, there exist differential operators $\mathcal{L}_{ij} \in \mathbb{K}[\partial]$ such that $H_i(U) = \sum_{j=1}^{n-1} \mathcal{L}_{ij}(u_j)$. If $B(X,U) = \sum_{i=1}^n \mathcal{F}_i(x_i - a_i)$ then $\sum_{i=1}^n \mathcal{F}_i(H_i(U)) = 0$. As a consequence, for each $j \in \{1,\ldots,n-1\}$ then $\sum_{i=1}^n \mathcal{F}_i(\mathcal{L}_{ij}(u_j)) = 0$. Let $\mathcal{L} = \mathrm{IDcont}(B)$ then $\mathcal{F}_i = \mathcal{L}\mathcal{L}_i$ with $\mathcal{L}_i \in \mathbb{K}[\partial]$ and $\mathrm{IDprim}(B) = \sum_{i=1}^n \mathcal{L}_i(x_i - a_i)$. Thus $\mathcal{L}\sum_{i=1}^n \mathcal{L}_i\mathcal{L}_{ij} = 0$ and $\mathcal{L} \neq 0$ so the differential operator $\sum_{i=1}^n \mathcal{L}_i\mathcal{L}_{ij} = 0$. We conclude that $\sum_{i=1}^n \mathcal{L}_i(H_i(U)) = 0$. Therefore $\mathrm{IDprim}(B) \in \mathbb{K}\{X\}$ which proofs the lemma. □

5 Conditions for $\dim(\mathrm{ID}) = n-1$

Let $\mathcal{P}(X,U)$, F_i, H_i be as in Section 2. Let $\mathrm{PS} = \mathrm{PS}(F_1,\ldots,F_n)$ and let ID be the implicit ideal of $\mathcal{P}(X,U)$. Let S and M_{L-1} be the leading and principal matrices of $\mathcal{P}(X,U)$ respectively, as defined in Section 3.1. Let $\mathcal{G}$ be the Groebner basis associated to the system $\mathcal{P}(X,U)$, $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$ and denote by $|\mathcal{G}_0|$ the number of elements of $\mathcal{G}_0$.

By Lemma 5.3 the ideal $(\mathrm{PS}) \cap \mathbb{K}\{X\} = (\mathcal{G}_0)$ is nonzero. Given a nonzero ID-primitive differential polynomial A in $(\mathrm{PS}) \cap \mathbb{K}\{X\}$ and co-order $\mathrm{c}(A)$ as defined in Section 4.1, in this section we provide necessary and sufficient conditions on S, $M(L)$ and A for $A(X) = 0$ to be the implicit equation of $\mathcal{P}(X, U)$.

5.1 Necessary conditions for $\dim(\mathrm{ID}) = n - 1$

For $N = \sum_{i=1}^{n} o_i = 0$ then $\mathcal{P}(X, U)$ in a system of n linear equations in $n - 1$ indeterminates.

Lemma 5.1. *If $N = 0$ then $\dim \mathrm{ID} = n - 1$ if and only if $\mathrm{rank}(S) = n - 1$.*

Proof. The matrix $M(L)$ is the $n \times n$ matrix whose $n \times (n-1)$ principal submatrix is S and whose last column contains $x_i - a_i$ in the i-th row, $i = 1, \ldots, n$. Then for linear $U_i \in \mathbb{K}\{X\}$

$$\mathrm{rank}(S) = n - 1 \Leftrightarrow \mathcal{G} = \{B_0, u_1 - U_1(X), \ldots, u_n - U_n(X)\}.$$

Equivalently $\{B_0\}$ is a characteristic set of ID. □

We prove in the next theorem that one statement is true in general.

Theorem 5.2. *Let $\mathcal{P}(X, U)$ be a system of linear DPPEs with implicit ideal ID and leading matrix S. If $\dim \mathrm{ID} = n - 1$ then $\mathrm{rank}(S) = n - 1$.*

Proof. For $N = 0$ the result was proved in Lemma 5.1. Let us suppose that $N \geq 1$. Let $\overline{\mathrm{PS}} = \{\partial^{N-o_i-\gamma-1} F_i, \ldots, \partial F_i, F_i \mid i = 1, \ldots, n\}$. Let $\mathcal{V}^h$ as defined in Section 3 and let $\overline{\mathcal{X}} = \{x_i, \ldots, x_{iN-o_i-\gamma-1} \mid i = 1, \ldots n\}$. Let M_{2L^h} be the $L^h \times (2L^h + 1)$ matrix whose k-th row contains the coefficients, as a polynomial in $\mathbb{K}[\overline{\mathcal{X}}][\mathcal{V}^h]$, of the $(L^h - k + 1)$-th polynomial in $\overline{\mathrm{PS}}$ and where the coefficients are written in decreasing order w.r.t. $\mathcal{R}^\star$. Then $\mathrm{rank}(M_{2L^h}) = L^h$ since M_{2L^h} has the $L^h \times L^h$ identity matrix as a submatrix.

Since $\mathrm{rank}(S) \leq n - 1$ then $\mathrm{Ker}(S^T) \neq \{\overline{0}\}$. Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathrm{Ker}(S^T)$, $\alpha \neq \overline{0}$ then

$$B(X, U) = \sum_{i=1}^{n} \alpha_i \partial^{N-o_i-\gamma} F_i(X, U) - \sum_{i=1}^{n} \alpha_i \partial^{N-o_i-\gamma}(x_i - a_i) \in \mathbb{K}[\overline{\mathcal{X}}][\mathcal{V}^h].$$

Since $\mathrm{rank}(M_{2L^h}) = L^h$ and the first L^h columns of M_{2L^h} correspond to the columns indexed by $\mathcal{V}^h$ then there exists $\mathcal{G}_i \in \mathbb{K}[\partial]$ with $\deg(\mathcal{G}_i) \leq N - o_i - \gamma - 1$, $i = 1, \ldots, n$ such that

$$B(X, U) + \sum_{i=1}^{n} \mathcal{G}_i(F_i(X, U)) \in \mathbb{K}[\overline{\mathcal{X}}].$$

Let

$$A_\alpha = \sum_{i=1}^{n} \alpha_i \partial^{N-o_i-\gamma} F_i(X,U) + \sum_{i=1}^{n} \mathcal{G}_i(F_i(X,U)).$$

Then $A_\alpha \in (PS) \cap \mathbb{K}\{X\}$ has leading vector $l(A_\alpha) = \alpha$ and co-order $\mathrm{c}(A_\alpha) = 0$.

Since $\dim \mathrm{ID} = n-1$ then $A_\alpha \in \mathrm{ID} = [A]$ for some differential polynomial $A \in \mathbb{K}\{X\}$ which is linear by Lemma 5.3. There exists $\mathcal{D}_\alpha \in \mathbb{K}[\partial]$ with $\deg(\mathcal{D}_\alpha) = \mathrm{c}(A)$ such that $A_\alpha = \mathcal{D}_\alpha(A)$. Then there is a nonzero $c_\alpha \in \mathbb{K}$ such that $l(A_\alpha) = c_\alpha l(A)$.

Given $\beta \in \mathrm{Ker}(S^T)$, $\overline{0} \neq \beta \neq \alpha$ there exist $A_\beta \in \mathrm{ID} = [A]$ with $l(A_\beta) = \beta$, $\mathrm{c}(A_\beta) = 0$ and a nonzero $c_\beta \in \mathbb{K}$ such that $l(A_\beta) = c_\beta l(A)$. Hence $\beta = (c_\beta / c_\alpha)\alpha$ which proves that the dimension of $\mathrm{Ker}(S^T)$ is equal to 1. Equivalently, $\mathrm{rank}(S) = n-1$. $\square$

As expected, for $N > 0$ the next results show that we need some more requirements for $\dim(\mathrm{ID}) = n-1$.

Lemma 5.3. *1. The number of elements of $\mathcal{G}_0$ equals $|\mathcal{G}_0| = L - \mathrm{rank}(M_{L-1})$.*

2. For every nonzero linear $B \in (\mathcal{G}_0)$ then $|\mathcal{G}_0| \geq \mathrm{c}(B) + 1$.

Proof. 1. Let M_{2L} be the $L \times 2L$ matrix defined in Section 4 and E_{2L} its redured echelon form. The number of elements of $\mathcal{G}_0$ is the number of rows in E_{2L} with zeros in the first $L-1$ columns. Thus $|\mathcal{G}_0| = L - \mathrm{rank}(M_{L-1})$.

2. Given $B \in (\mathcal{G}_0) = (\mathrm{PS}) \cap \mathbb{K}\{X\}$. By definition of $\mathrm{c}(B)$ then $\partial B, \ldots, \partial^{\mathrm{c}(B)} B \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$. Also, there exists $k \in \{1, \ldots, n\}$ such that $\mathrm{ord}(B, x_k) = N - o_k - \gamma - \mathrm{c}(B)$, we can assume that the coefficient of $x_{kN-o_k-\gamma}$ in B is 1. Thus M_{2L} is row equivalent to an $L \times 2L$ matrix with $\partial^{\mathrm{c}(B)} B, \ldots, \partial B, B$ in the last $\mathrm{c}(B)+1$ rows. Namely, replace the row of M_{2L} corresponding to the coefficients of $\partial^{N-o_k-\gamma-t} F_k$ by $\partial^{\mathrm{c}(B)-t} B$, $t = 0, \ldots, \mathrm{c}(B)$ and next reorder the rows of the obtained matrix. Therefore $|\mathcal{G}_0| \geq \mathrm{c}(B) + 1$.

$\square$

If the dimension of ID is $n-1$ then $\mathrm{ID} = [A]$ for some $A \in (\mathrm{PS}) \cap \mathbb{K}\{X\} = (\mathcal{G}_0)$ and $\{A\}$ is a characteristic set of ID. By Lemma 5.3 then A is linear and we give some more requirements for A in the next theorem.

Theorem 5.4. *Let $\mathcal{G}$ be the Groebner basis associated to the system $\mathcal{P}(X,U)$ with implicit ideal* ID*, $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. If $\dim \mathrm{ID} = n-1$ then $\mathrm{ID} = [A]$ where A is a nonzero linear differential polynomial verifying.*

1. *A is an* ID*-primitive differential polynomial in $(\mathcal{G}_0)$.*

2. $|\mathcal{G}_0| = \mathrm{c}(A) + 1$.

Proof. 1. Let $\mathcal{L} = \mathrm{IDcont}(A)$ and $A' = \mathrm{IDprim}(A)$. If A is not ID-primitive then $\deg(\mathcal{L}) \geq 1$ and $A = \mathcal{L}(A')$ contradicting that $\{A\}$ is a characteristic set of ID.

2. If $\dim \mathrm{ID} = n-1$ and $|\mathcal{G}_0| = m+1$ then $\mathcal{G}_0 = \{B_0, \mathcal{D}_1(B_0), \ldots, \mathcal{D}_m(B_0)\}$ with $\mathcal{D}_i \in \mathbb{K}[\partial]$, $\deg(\mathcal{D}_1) > 0$, $\deg(\mathcal{D}_{i+1}) > \det(\mathcal{D}_i)$, $i = 1, \ldots, m$. Then $m \leq \mathrm{c}(B_0)$. Thus by Lemma 5.3 then $|\mathcal{G}_0| = \mathrm{co}(B_0) + 1$. On the other hand $\mathrm{ID} = [B_0]$ and $A \in \mathrm{ID}$ then $A = \mathcal{D}(B_0)$ for some $\mathcal{D} \in \mathbb{K}[\partial]$ but we just proved that A is ID-primitive then $\mathcal{D} \in \mathbb{K}$. This implies $|\mathcal{G}_0| = \mathrm{c}(A) + 1$.

□

Observe that, if $\dim \mathrm{ID} = n-1$, given A and B nonzero linear ID-primitive differential polynomials in $(\mathcal{G}_0)$ then $\mathrm{ID} = [A] = [B]$ and $\mathrm{c}(A) = \mathrm{c}(B)$.

Corollary 5.5. *Let $\mathcal{G}$ be the Groebner basis associated to the system $\mathcal{P}(X,U)$ with implicit ideal* ID, *$\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. If $\dim \mathrm{ID} = n-1$, for every nonzero linear* ID*-primitive differential polynomial A in $(\mathcal{G}_0)$ then $\mathrm{ID} = [A]$ and $|\mathcal{G}_0| = \mathrm{c}(A) + 1$.*

5.2 Sufficient conditions for $\dim(\mathrm{ID}) = n - 1$

Given a nonzero linear ID-primitive differential polynomial A in $(\mathrm{PS}) \cap \mathbb{K}\{X\}$ we give sufficient conditions for A to be a characteristic polynomial of ID.

Lemma 5.6. *Let $\mathcal{G}$ be the Groebner basis associated to the system $\mathcal{P}(X,U)$ with implicit ideal* ID, *$\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. Let S be the principal matrix of $\mathcal{P}(X,U)$. Given a nonzero linear* ID*-primitive differential polynomial A in $(\mathcal{G}_0)$ with $|\mathcal{G}_0| = \mathrm{c}(A) + 1$ the following statements hold.*

1. *For $j = 0, 1, \ldots, \mathrm{c}(A)$ there exist $\mathcal{D}_j \in \mathbb{K}[\partial]$ with $\deg(\mathcal{D}_j) = j$ such that $\mathcal{G}_0 = \{B_0 = \mathcal{D}_0(A), B_1 = \mathcal{D}_1(A), \ldots, B_{\mathrm{c}(A)} = \mathcal{D}_{\mathrm{c}(A)}(A)\}$.*

2. *If $\mathrm{rank}(S) = n-1$. Given $B \in \mathcal{G} \backslash \mathcal{G}_0$, let us suppose there exists a positive integer e_B such that $1 \leq e_B \leq \mathrm{c}(A)$ and $\mathrm{ord}(B, u_j) \leq N - \gamma_j - \gamma - e_B$, $j = 1, \ldots, n-1$. Then there exists $\overline{B} \in (\mathcal{G}_0)$ such that $\mathrm{c}(B - \overline{B}) \geq e_B$.*

Proof. 1. Since $|\mathcal{G}_0| = L - \mathrm{rank}(M_{L-1}) = \mathrm{c}(A) + 1$, there exists an echelon form E of M_{2L} whose last $\mathrm{c}(A) + 1$ rows contain the coefficients of $\partial^{\mathrm{c}(A)} A, \ldots, \partial A, A$. Then the last $\mathrm{c}(A) + 1$ rows of the reduced echelon form E_{2L} of M_{2L} contain the coefficients of $B_{\mathrm{c}(A)} = \mathcal{D}_{\mathrm{c}(A)}(A), \ldots, B_1 = \mathcal{D}_1(A), B_0 = \mathcal{D}_0(A)$ for some $\mathcal{D}_j \in \mathbb{K}[\partial]$ with $\deg(\mathcal{D}_j) = j$ where $j = 0, 1, \ldots, \mathrm{c}(A)$. Then $\mathcal{G}_0 = \{B_0, B_1, \ldots, B_{\mathrm{c}(A)}\}$.

2. Let $s \in \{0, \ldots, e_B - 1\}$. Then by 1 the co-order of $B_{\mathrm{c}(A)-s}$ equals $\mathrm{c}(B_{\mathrm{c}(A)-s}) = s$. Since $B_{\mathrm{c}(A)-s} \in \mathbb{K}\{X\}$ then $\mathrm{ord}(B_{\mathrm{c}(A)-s}, u_j) < N - \gamma_j - \gamma - s$, hence by Remark 4.4 then $\mathrm{l}(B_{\mathrm{c}(A)-s}) \in \mathrm{Ker}(S^T)$. Given $B \in \mathcal{G}\backslash\mathcal{G}_0$ we will prove by induction on s that for $s = 0, \ldots, e_B - 1$ there exists $C_s \in (\mathcal{G}_0)$ such that $\mathrm{c}(B - C_s) \geq s + 1$. Then $\overline{B} = C_{e_B - 1}$.

   There exists $\mathcal{G}_i \in \mathbb{K}[\partial]$ with $\deg(\mathcal{G}_i) \leq N - o_i - \gamma$, $i = 1, \ldots, n$ such that $B = \sum_{i=1}^{n} \mathcal{G}_i(F_i(X, U))$. Let β_i be the coefficient of $\partial^{N-o_i-\gamma}$ in $\mathcal{G}_i$. By assumption $\mathrm{ord}(B, u_j) \leq N - \gamma_j - \gamma - e_B < N - \gamma_j - \gamma$ so $\beta = (\beta_1, \ldots, \beta_n) \in \mathrm{Ker}(S^T)$. Now $\mathrm{rank}(S) = n - 1$ which means that $\dim \mathrm{Ker}(S^T) = 1$ so there exists $\mu \in \mathbb{K}$ such that $\beta = \mu l(B_{\mathrm{c}(A)})$. Let $C_0 = \mu B_{\mathrm{c}(A)}$ then $\mathrm{c}(B - C_0) \geq 1$. This proves the claim for $s = 0$.

   Assuming the claim is true for $s - 1$, $s \geq 1$ then there exists $C_{s-1} \in (\mathcal{G}_0)$ such that $\mathrm{c}(B - C_{s-1}) \geq s$. Then $B - C_{s-1} = \sum_{i=1}^{n} \mathcal{G}_i^s(F_i(X, U))$ where $\mathcal{G}_i^s \in \mathbb{K}[\partial]$ with $\deg(\mathcal{G}_i^s) \leq N - o_i - \gamma - s$, $i = 1, \ldots, n$. Let β_i^s be the coefficient of $\partial^{N-o_i-\gamma-s}$ in $\mathcal{G}_i^s$. By assumption $\mathrm{ord}(B - C_{s-1}, u_j) \leq N - \gamma_j - \gamma - e_B < N - \gamma_j - \gamma - s$ so $\beta^s = (\beta_1^s, \ldots, \beta_n^s) \in \mathrm{Ker}(S^T)$. Now there exists $\mu_s \in \mathbb{K}$ such that $\beta^s = \mu_s l(B_{\mathrm{c}(A)-s})$. Let $C_s = C_{s-1} + \mu_s B_{\mathrm{c}(A)-s}$ then $\mathrm{c}(B - C_s) \geq s + 1$.

□

Theorem 5.7. *Let $\mathcal{G}$ be the Groebner basis associated to the system $\mathcal{P}(X, U)$ with implicit ideal* ID, *$\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. Let S be the principal matrix of $\mathcal{P}(X, U)$. If* $\mathrm{rank}(S) = n - 1$, *for every nonzero linear* ID*-primitive differential polynomial A in $(\mathcal{G}_0)$ with $|\mathcal{G}_0| = \mathrm{c}(A) + 1$ then A is a characteristic polynomial of* ID.

Proof. If $\mathrm{c}(A) = 0$ then $\mathrm{rank}(M_{L-1}) = L - 1$. By Theorem 18(2) and Lemma 20(4) in [13] and Lemma 5.6(1) then A is a characteristic polynomial of ID.

Let us suppose that $\mathrm{c}(A) > 0$. For $j \in \{1, \ldots, n - 1\}$ let

$$\mathcal{G}_j = \{B \in \mathcal{G} \mid \mathrm{lead}(B) = u_{jk}, k \in \{0, 1, \ldots, N - \gamma_j - \gamma\}\}.$$

Then $\mathcal{G}\backslash\mathcal{G}_0 = \cup_{j=1}^{n-1} \mathcal{G}_j$. We use Algorithm 4.1 to prove the result. Set $\mathcal{A}^0 = \{B_0\}$ then by Lemma 5.6(1) we have $\mathrm{prem}(B_i, \mathcal{A}^{i-1}) = 0$, $i = 1, \ldots, \mathrm{c}(A)$, thus $\mathcal{A}^i = \mathcal{A}^0$. Given $i \in \{\mathrm{c}(A) + 1, \ldots, L - 1\}$ then $B_i \in \mathcal{G}_{j_i}$ for some $j_i \in \{1, \ldots, n - 1\}$ and if $\mathcal{G}_{j_i} \cap \mathcal{A}^{i-1} = \emptyset$ then $\mathrm{prem}(B_i, \mathcal{A}^{i-1}) \notin \mathbb{K}\{X\}$ and $\mathcal{A}^i = \mathcal{A}^{i-1} \cup \{\mathrm{prem}(B_i, \mathcal{A}^{i-1})\}$. If $i = \mathrm{c}(A) + 1$ then $\mathcal{G}_{j_i} \cap \mathcal{A}^{i-1} = \emptyset$, we will prove by induction on i that for $i = \mathrm{c}(A) + 2, \ldots, L - 1$ if $\mathcal{G}_{j_i} \cap \mathcal{A}^{i-1} \neq \emptyset$ then $\mathrm{prem}(B_i, \mathcal{A}^{i-1}) = 0$ and so $\mathcal{A}^i = \mathcal{A}^{i-1}$. This proves that $\mathcal{A}_0 = \mathcal{A}^0$ and A is a characteristic polynomial of ID.

For $i = \mathrm{c}(A) + 2$ let us suppose that $\mathcal{G}_{j_i} \cap \mathcal{A}^{i-1} \neq \emptyset$ then $|\mathcal{G}_0| = L - \mathrm{rank}(M_{L-1}) = \mathrm{c}(A) + 1$ implies

$$1 \leq e_i = \mathrm{ord}(B_i, u_{j_i}) - \mathrm{ord}(B_{i-1}, u_{j_i}) \leq \mathrm{c}(A).$$

Hence $\mathrm{ord}(B_{i-1}, u_{j_i}) \leq N - \gamma_{j_i} - \gamma - e_i$ and by Lemma 5.6(2) there exists $\overline{B_i} \in (\mathcal{G}_0)$ such that $\mathrm{c}(B_{i-1} - \overline{B_i}) \geq e_i$. Thus $\partial^t(B_{i-1} - \overline{B_i}) \in (\mathrm{PS})$ for $t = 1, \ldots, e_i$. Therefore $C_i = \mathrm{prem}(B_i, B_{i-1} - \overline{B_i}) \in (\mathrm{PS})$ with $\mathrm{lead}(C_i) < \mathrm{lead}(B_{i-1})$ so $C_i \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$. Since $\mathrm{prem}(B_{i-1} - \overline{B_i}, \mathcal{A}^{i-1}) = 0$ then $\mathrm{prem}(B_i, \mathcal{A}^{i-1}) = \mathrm{prem}(C_i, \mathcal{A}^{i-1}) = \mathrm{prem}(C_i, \mathcal{A}^0) = 0$.

Given $i \in \{\mathrm{c}(A)+3, \ldots, L-1\}$ with $\mathcal{G}_{j_i} \cap \mathcal{A}^{i-1} \neq \emptyset$ then there exists $B \in \{B_{\mathrm{c}(A)+1}, \ldots, B_{i-1}\}$ such that

$$1 \leq e_B = \mathrm{ord}(B_i, u_{j_i}) - \mathrm{ord}(B, u_{j_i}) \leq \mathrm{c}(A).$$

Hence $\mathrm{ord}(B, u_{j_i}) \leq N - \gamma_{j_i} - \gamma - e_B$ and by Lemma 5.6(2) there exists $\overline{B} \in (\mathcal{G}_0)$ such that $\mathrm{c}(B - \overline{B}) \geq e_B$. Thus $\partial^{e_B}(B - \overline{B}) \in (\mathrm{PS})$. Therefore $C_i = B_i - \partial^{e_B}(B_{i-1} - \overline{B}) \in (\mathrm{PS})$ with $\mathrm{lead}(C_i) < \mathrm{lead}(B_i)$ then $C_i = \gamma_0 B_0 + \cdots + \gamma_{i-1} B_{i-1}$ with $\gamma_0, \ldots, \gamma_{i-1} \in \mathbb{K}$. By induction on i then

$$\mathrm{prem}(B_{i-1} - \overline{B}, \mathcal{A}^{i-1}) = \mathrm{prem}(\mathrm{prem}(B_{i-1} - \overline{B}, \mathrm{prem}(B_{i-1}, \mathcal{A}^{i-2})), \mathcal{A}^{i-2}) = 0$$

so we have $\mathrm{prem}(B_i, \mathcal{A}^{i-1}) = \mathrm{prem}(C_i, \mathcal{A}^{i-1})$. By induction we also have $\mathrm{prem}(B_0, \mathcal{A}^{i-1}) = 0, \ldots, \mathrm{prem}(B_{i-1}, \mathcal{A}^{i-1}) = 0$ then $\mathrm{prem}(C_i, \mathcal{A}^{i-1}) = 0$. $\square$

Corollary 5.8. *Given a system $\mathcal{P}(X,U)$ of linear DPPEs with implicit ideal* ID. *Let S and M_{L-1} be the leading and principal matrices of $\mathcal{P}(X,U)$ respectively. The following statements are equivalent.*

1. *The dimension of* ID *is $n-1$.*
2. $\mathrm{rank}(S) = n-1$ *and there exists a nonzero linear* ID*-primitive differential polynomial A such that $L - \mathrm{rank}(M_{L-1}) = \mathrm{c}(A) + 1$.*
3. $\mathrm{rank}(S) = n-1$ *and for every nonzero linear* ID*-primitive differential polynomial B then $L - \mathrm{rank}(M_{L-1}) = \mathrm{c}(B) + 1$.*

In such situation $A(X) = 0$ is the implicit equation of $\mathcal{P}(X,U)$.

Proof. By Theorem 5.2 and Theorem 5.4 then (1)⇒(2) and by Corollary 5.5 (1)⇒(3). Theorem 5.7 proves (3)⇒(1). $\square$

6 Linear perturbations of $\mathcal{P}(X,U)$

Let $\mathcal{P}(X,U)$, F_i, H_i be as in Section 2. Let p be a differential indeterminate over $\mathbb{K}$ such that $\partial(p) = 0$. Denote by $\mathbb{K}_p = \mathbb{K}\langle p \rangle$ the differential field extension of $\mathbb{K}$ by p. A linear perturbation of the system $\mathcal{P}(X,U)$ is a new system

$$\mathcal{P}_\phi(X,U) = \begin{cases} x_1 &= P_1(U) + p\,\phi_1(U) \\ & \vdots \\ x_n &= P_n(U) + p\,\phi_n(U). \end{cases}$$

where the linear perturbation $\phi = (\phi_1(U), \ldots, \phi_n(U))$ is a family of linear differential polynomials in $\mathbb{K}\{U\}$. For $i = 1, \ldots, n$ let

$$F_i^\phi(X,U) = F_i(X,U) - p\,\phi_i(U) \text{ and } H_i^\phi(U) = H_i(U) - p\,\phi_i(U).$$

Let $\mathrm{PS}_\phi = \mathrm{PS}(F_1^\phi, \ldots, F_n^\phi)$, a set of linear differential polynomials in $\mathbb{K}_p[\mathcal{X}][\mathcal{V}] \subset \mathbb{K}_p\{X \cup U\}$ and let (PS_ϕ) be the ideal generated by PS_ϕ in $\mathbb{K}_p[\mathcal{X}][\mathcal{V}]$. We prove next the existence of a linear perturbation ϕ such that $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi) \neq 0$.

Let us suppose that $o_n \geq o_{n-1} \geq \ldots \geq o_1$ to define the perturbation $\phi = (\phi_1(U), \ldots, \phi_n(U))$ by

$$\phi_i(U) = \begin{cases} \varepsilon_i u_{n-i-1, o_i - \gamma_{n-i-1}} + u_{n-i}, & i = 1, \ldots, n-2, \\ u_1, & i = n-1, \\ \varepsilon_n u_{n-1, o_n - \gamma_{n-1}}, & i = n, \end{cases} \tag{4}$$

where $\varepsilon_i = 1$ if $o_i \neq 0$ and $\varepsilon_i = 0$ if $o_i = 0$, for $i = 1, \ldots, n$.

Let us suppose that $N \geq 1$. We denote by $M_\phi(L^h)$ the complete differential homogeneous resultant matrix for the set of linear differential polynomials $H_1^\phi, \ldots, H_n^\phi$. Then $M_\phi(L^h)$ is an $L^h \times L^h$ matrix with elements in $\mathbb{K}[p]$ and there exists an $L^h \times L^h$ matrix M_ϕ with elements in $\mathbb{K}$ such that $M_\phi(L^h) = M(L^h) - p\,M_\phi$. Then

$$\partial\mathrm{CRes}^h(H_1^\phi, \ldots, H_n^\phi) = \det(M_\phi(L^h)) = \det(M(L^h) - p\,M_\phi).$$

As expected, the linear perturbation that makes $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi) \neq 0$ is not unique. Our proposal can be viewed as a generalization of the characteristic polynomial of a matrix to the linear differential case in the spirit of [2]. Namely, for $o_n = 1$, $o_{n-1} = \cdots = o_1 = 0$ then we obtain the characteristic polynomial of $M(L^h)$, namely $\partial\mathrm{CRes}^h(H_1 - p\,\phi_1, \ldots, H_n - p\,\phi_n) = \det(M(L^h) - p\,I_{L^h})$ where I_{L^h} is the $L^h \times L^h$ identity matrix.

Let S^ϕ be the leading matrix of $\mathcal{P}_\phi(X,U)$. For $i \in \{1, \ldots n\}$ let S_i^ϕ be the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row of S^ϕ.

Proposition 6.1. *Given a system $\mathcal{P}(X,U)$ of linear DPPEs and the perturbation ϕ defined by (4). The following statements hold.*

1. *The determinant of S_n^ϕ is nonzero and it has degree $n-1$ in p.*

2. *If $N \geq 1$ then the linear complete homogeneous differential resultant $\partial\mathrm{CRes}^h(H_1^\phi, \ldots, H_n^\phi)$ is a polynomial in $\mathbb{K}[p]$ of degree L^h and not identically zero.*

Proof. 1. Observe that S_n^ϕ has p's in the main diagonal.

2. We have $\det(M_\phi(L^h)) = p^{L^h}\det((1/p)M(L^h) - M_\phi)$. If we set $y = 1/p$ the matrix obtained from $yM(L^h) - M_\phi$ at $y = 0$ is M_ϕ. We will prove that $\det(M_\phi) \neq 0$ and therefore the degree of $\det(M_\phi(L^h))$ in p is L^h.

   In each row and column of M_ϕ there is at least one nonzero entry and at most two. Also, if a column has two nonzero entries at least one of them is the only nonzero entry of its row. Therefore we can reorganize the rows of M_ϕ to get a matrix N which has ones in the main diagonal and in every row at most one nonzero entry not in the main diagonal. Namely the i-th row of N is either the only row of M_ϕ with a nonzero entry in the i-th column or the row of M_ϕ with its only nonzero entry in the i-th column. Thus N is a product of elementary matrices and so it has nonzero determinant.

□

Theorem 6.2. *Given a system $\mathcal{P}(X,U)$ of linear DPPEs there exists a linear perturbation ϕ such that the differential resultant $\partial\mathrm{CRes}(F_1^\phi,\ldots,F_n^\phi)$ is a nonzero polynomial in $\mathbb{K}[p]\{X\}$ and $\det(S_n^\phi)\neq 0$.*

Proof. Let ϕ be the perturbation defined by (4) then $\det(S_n^\phi)\neq 0$ by Proposition 6.1. If $N = 0$ the result follows from

$$\partial\mathrm{CRes}(F_1^\phi,\ldots,F_n^\phi) = \sum_{i=1}^{n}(-1)^{i+n}\det(S_i^\phi)(x_i - a_i).$$

If $N \geq 1$ then $\partial\mathrm{CRes}^h(H_1^\phi,\ldots,H_n^\phi)\neq 0$ by Proposition 6.1. This is equivalent by [13], Theorem 18(2) to $\partial\mathrm{CRes}(F_1^\phi,\ldots,F_n^\phi)\neq 0$. □

If nonzero then $\partial\mathrm{CRes}(F_1^\phi,\ldots,F_n^\phi)$ is a polynomial in p whose coefficients are linear differential polynomials in $\mathbb{K}\{X\}$. We focus our attention next in the coefficient of the lowest degree term in p of $\partial\mathrm{CRes}(F_1^\phi,\ldots,F_n^\phi)$.

Theorem 6.3. *Given a system $\mathcal{P}(X,U)$ of linear DPPEs let ϕ be a linear perturbation such that $\partial\mathrm{CRes}(F_1^\phi,\ldots,F_n^\phi)\neq 0$ and $\det(S_n^\phi)\neq 0$. The following statements hold.*

1. *There exists a linear differential polynomial P in $(\mathrm{PS}_\phi)\cap\mathbb{K}_p\{X\}$ with coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$ such that $\alpha_n \neq 0$ where $l(P) = (\alpha_1,\ldots,\alpha_n)$.*

2. *There exists $a\in\mathbb{N}$ such that*

$$\alpha_n\partial\mathrm{CRes}(F_1^\phi,\ldots,F_n^\phi) = (-1)^a\det(S_n^\phi)\partial\mathrm{CRes}^h(H_1^\phi,\ldots,H_n^\phi)P(X). \quad (5)$$

   Furthermore $\frac{\det(S_n^\phi)\partial\mathrm{CRes}^h(H_1^\phi,\ldots,H_n^\phi)}{\alpha_n}\in\mathbb{K}[p]$.

Proof. 1. By Lemma 5.3 there exists a nonzero linear differential polynomial $B \in (\mathrm{PS}_\phi) \cap \mathbb{K}_p\{X\}$. There exists $c \in \mathbb{K}_p$ such that $P(X) = c\partial^{\mathrm{c}(B)} B(X)$ has coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$. Observe that $\mathrm{co}(P) = 0$. Since $P \in \mathbb{K}_p\{X\}$ then $\mathrm{ord}(P, u_j) < N - \gamma_j - \gamma$ and by Remark 4.4 it holds $l(B) \in \mathrm{Ker}((S^\phi)^T)$. If $\alpha_n = 0$ then $\det(S_n^\phi) \neq 0$ implies $\alpha_i = 0$, $i = 1, \ldots, n$. This contradicts that $\mathrm{c}(P) = 0$ therefore $\alpha_n \neq 0$.

2. Equation (5) follows from [13], Theorem 18(1). Since the content of $P(X)$ belongs to $\mathbb{K}$ then α_n divides $\det(S_n^\phi)\partial\mathrm{CRes}^h(H_1^\phi, \ldots, H_n^\phi)$ in $\mathbb{K}[p]$. □

Remark 6.4. *In the situation of the previous theorem and with the perturbation ϕ defined by* (4) *we can add the following remarks. For $i = 1, \ldots, n$ by formula* (3) *then $\det(S_i^\phi)\partial\mathrm{CRes}^h(H_1^\phi, \ldots, H_n^\phi)$ is the coefficient of $x_{iN-o_i-\gamma}$ in $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi)$. By 1 and 2 in the previous theorem then*

$$\alpha_n \det(S_i^\phi)\partial\mathrm{CRes}^h(H_1^\phi, \ldots, H_n^\phi) = \det(S_n^\phi)\partial\mathrm{CRes}^h(H_1^\phi, \ldots, H_n^\phi)\alpha_i,$$

hence $\det(S_i^\phi) = \det(S_n^\phi)\alpha_i/\alpha_n \in \mathbb{K}[p]$. Let $\alpha = \gcd(\alpha_n, \det(S_n^\phi))$. If $\alpha \in \mathbb{K}$ then the degree of $\det(S_i^\phi)$ in p is greater or equal than $n-2$ which is not possible then $\gcd(\alpha_n, \det(S_n^\phi)) \in \mathbb{K}[p]\backslash\mathbb{K}$.

Let D_ϕ be the lowest degree of p in $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi)$ and let A_{D_ϕ} be the coefficient of p^{D_ϕ} in $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi)$. We call D_ϕ the degree of the perturbed system $\mathcal{P}_\phi(X, U)$. Observe that

$$D_\phi = 0 \Leftrightarrow \partial\mathrm{CRes}(F_1, \ldots, F_n) \neq 0.$$

We write $D_\phi = -1$ if $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi) = 0$ and so

$$D_\phi \geq 0 \Leftrightarrow \partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi) \neq 0.$$

In the remaining parts of this section we assume that $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi) \neq 0$. Let $\mathrm{PS} = \mathrm{PS}(F_1, \ldots, F_n)$ and let ID be the implicit ideal of $\mathcal{P}(X, U)$. We will use $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi)$ to provide a nonzero ID-primitive differential polynomial A_ϕ in $(\mathrm{PS}) \cap \mathbb{K}\{X\}$.

Lemma 6.5. *The linear differential polynomial A_{D_ϕ} belongs to* $(\mathrm{PS}) \cap \mathbb{K}\{X\}$.

Proof. By [13], Proposition 16 then $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi) \in (\mathrm{PS}_\phi) \cap \mathbb{K}_p\{X\}$ and it equals $p^{D_\phi}(A_{D_\phi} + pA')$ for some $A' \in \mathbb{K}_p\{X\}$. Therefore the linear polynomial $A_{D_\phi} + pA' \in (\mathrm{PS}_p) \cap \mathbb{K}_p\{X\}$ and there exist $\mathcal{F}_i \in \mathbb{K}_p[\partial]$ with $\deg(\mathcal{F}_i) \leq N - o_i - \gamma$ such that $A_{D_\phi}(X) + pA'(X) = \sum_{i=1}^n \mathcal{F}_i(F_i^\phi(X, U))$. Then $A_{D_\phi}(X) + pA'(X) = \sum_{i=1}^n \mathcal{F}_i(x_i - a_i)$ and $\sum_{i=1}^n \mathcal{F}_i(H_i^\phi(U)) = 0$.

For each $i \in \{1,\dots,n\}$ there exists $\mathcal{L}_i \in \mathbb{K}[\partial]$ and $\mathcal{F}'_i \in \mathbb{K}_p[\partial]$ such that $\mathcal{F}_i = \mathcal{L}_i + p\mathcal{F}'_i$. Then $A_{D_\phi}(X) + pA'(X) = \sum_{i=1}^n \mathcal{L}_i(x_i - a_i) + p\sum_{i=1}^n \mathcal{F}'_i(x_i - a_i)$ and hence $A_{D_\phi}(X) = \sum_{i=1}^n \mathcal{L}_i(x_i - a_i)$. On the other hand

$$0 = \sum_{i=1}^n \mathcal{F}_i(H_i^\phi(U)) = \sum_{i=1}^n \mathcal{L}_i(H_i(U)) + p\sum_{i=1}^n \mathcal{L}_i(\phi_i(U)) + p\sum_{i=1}^n \mathcal{F}'_i(H_i^\phi(U))$$

which implies $\sum_{i=1}^n \mathcal{L}_i(H_i(U)) = 0$. Thus $A_{D_\phi}(X) = \sum_{i=1}^n \mathcal{L}_i(F_i(X,U))$ with $\deg(\mathcal{L}_i) \le N - o_i - \gamma$ then $A_{D_\phi} \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$. □

Let A_ϕ be the ID-primitive part of A_{D_ϕ}. We call A_ϕ the differential polynomial associated to $\mathcal{P}_\phi(X,U)$. We relate D_ϕ with $\mathrm{c}(A_\phi)$ and give conditions on D_ϕ for $A_\phi(X) = 0$ to be the implicit equation of $\mathcal{P}(X,U)$.

Remark 6.6. *Let $\mathcal{P}_\phi(X,U)$ and $\mathcal{P}_\psi(X,U)$ be two different linear perturbations of $\mathcal{P}(X,U)$ with degrees $D_\phi \ge 0$ and $D_\psi \ge 0$. Let A_ϕ and A_ψ be the associated differential polynomials.*

1. *As illustrated in Example 1 of Section 7 the degrees D_ϕ and D_ψ may be different.*

2. *If $\dim \mathrm{ID} = n-1$ then $A_\phi = \gamma A_\psi$ for some $\gamma \in \mathbb{K}$.*

Theorem 6.7. *Let $\mathcal{P}_\phi(X,U)$ be a perturbed system of the system $\mathcal{P}(X,U)$ of degree $D_\phi \ge 0$. Let $\mathcal{G}$ be the Groebner basis associated to $\mathcal{P}(X,U)$ and $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{K}\{X\}$. Then $|\mathcal{G}_0| - 1 \le D_\phi$.*

Proof. Let $M_\phi(L)$ be the differential resultant matrix of $F_1^\phi, \dots, F_n^\phi$ and M_{L-1}^ϕ the $L \times (L-1)$ principal submatrix of $M_\phi(L)$. By equation (2) then

$$\partial\mathrm{CRes}(F_1^\phi, \dots, F_n^\phi) = \sum_{i=1}^n \sum_{k=0}^{N-o_i-\gamma} b_{ik} \det(M_{x_{ik}}^\phi)(x_{ik} - \partial^k a_i),$$

with $M_{x_{ik}}^\phi$ an $(L-1)\times(L-1)$ submatrix of M_{L-1}^ϕ and $b_{ik} = \pm 1$ according to the row index of $x_{ik} - \partial^k a_i$ in the matrix $M_\phi(L)$. Let $M_{x_{ik}}$ be the $(L-1)\times(L-1)$ submatrix of the principal matrix M_{L-1} of $\mathcal{P}(X,U)$. Let $r_{ik} = \mathrm{rank}(M_{x_{ik}})$, then there exists an invertible matrix E_{ik} of order $L-1$ and entries in $\mathbb{K}$ such that the last $L-1-r_{ik}$ rows of $E_{ik}M_{x_{ik}}$ are zero. If we divide each one of the last $L-1-r_{ik}$ rows of $E_{ik}M_{x_{ik}}^\phi$ by p we obtain a matrix N_{ik}^ϕ such that

$$\det M_\phi(L) = \sum_{i=1}^n \sum_{k=0}^{N-o_i-\gamma} b_{ik} p^{L-1-r_{ik}} \det(N_{ik}^\phi)(x_{ik} - \partial^k a_i).$$

This proves that $L-1-r_{ik} \leq D_\phi$ for all $i = 1, \ldots, n$ and $k = 0, 1, \ldots, N-o_i-\gamma$. Now $\mathrm{rank}(M_{L-1}) \geq r_{ik}$ so

$$|\mathcal{G}_0| - 1 = L - \mathrm{rank}(M_{L-1}) - 1 \leq L - 1 - r_{ik} \leq D_\phi.$$

□

We showed that $D_\phi \geq |\mathcal{G}_0| - 1 = L - \mathrm{rank}(M_{L-1}) - 1 \geq \mathrm{c}(A_\phi)$ and in general equality does not hold (see examples in Section 7).

Corollary 6.8. *Let $\mathcal{P}(X,U)$ be a system of linear DPPEs with implicit ideal* ID *and leading matrix S. Let $\mathcal{P}_\phi(X,U)$ be a perturbed system of $\mathcal{P}(X,U)$ of degree $D_\phi \geq 0$. Let A_ϕ be the differential polynomial associated to $\mathcal{P}_\phi(X,U)$. If* $\mathrm{rank}(S) = n-1$ *and $D_\phi = \mathrm{c}(A_\phi)$ then* ID *has dimension $n-1$ and $A_\phi(X) = 0$ is the implicit equation of $\mathcal{P}(X,U)$.*

Proof. If $D_\phi = \mathrm{c}(A_\phi)$ then by Theorem 6.7 we have $|\mathcal{G}_0| \leq \mathrm{c}(A_\phi) + 1$. By Lemma 5.3 then $|\mathcal{G}_0| = \mathrm{c}(A_\phi) + 1$ and by Corollary 5.8 the result follows. □

7 Implicitization algorithm for linear DPPEs and examples

Let $\mathcal{P}(X,U)$ be a system of linear DPPEs with implicit ideal ID. In this section, we give an algorithm that decides whether the dimension of ID is $n-1$ and in the affirmative case returns the implicit equation of $\mathcal{P}(X,U)$. For this purpose let S and M_{L-1} be the leading and principal matrices of $\mathcal{P}(X,U)$ respectively. Let $\mathcal{P}_\phi(X,U)$ be a perturbed system of $\mathcal{P}(X,U)$ of degree $D_\phi \geq 0$. Let A_{D_ϕ} be the coefficient of p^{D_ϕ} in $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi)$ and A_ϕ the differential polynomial associated to $\mathcal{P}_\phi(X,U)$.

Algorithm 7.1.
- Given *the system $\mathcal{P}(X,U)$ of linear DPPEs.*
- Decide *whether the dimension is $n-1$ and in the affirmative case*
- Return *a characteristic polynomial of* ID.

1. *Compute* $\mathrm{rank}(S)$.
2. *If* $\mathrm{rank}(S) < n-1$ RETURN "dimension less than $n-1$".
3. *Compute $\mathcal{P}_\phi(X,U)$ with perturbation ϕ given by* (4).
4. *Compute $\partial\mathrm{CRes}(F_1^\phi, \ldots, F_n^\phi)$, D_ϕ and A_{D_ϕ}.*
5. *If $D_\phi = 0$* RETURN A_{D_ϕ}.

6. *Compute* A_ϕ *and* $\mathrm{c}(A_\phi)$.
7. *If* $D_\phi = \mathrm{c}(A_\phi)$ RETURN A_ϕ.
8. *Compute* $\mathrm{rank}(M_{L-1})$.
9. *If* $L - \mathrm{rank}(M_{L-1}) > \mathrm{c}(A_\phi) + 1$ RETURN *"dimension less than $n-1$"*.
10. *If* $L - \mathrm{rank}(M_{L-1}) = \mathrm{c}(A_\phi) + 1$ RETURN A_ϕ.

The next examples were computed in Maple. The computation of differential resultants was carried out with our Maple implementation of the linear complete differential resultant, available at [11].

7.1 Example 1

Let $\mathbb{K} = \mathbb{Q}$, $\partial = \frac{\partial}{\partial t}$ and consider the system $\mathcal{P}(X,U)$ of linear DPPEs providing the set of differential polynomials in $\mathbb{K}\{x_1,x_2,x_3\}\{u_1,u_2\}$,

$$\begin{aligned}
F_1(X,U) &= x_1 + u_1 - u_2 + u_{11} - u_{12} - 4u_{21} - 3u_{22},\\
F_2(X,U) &= x_2 + u_2 + u_{11} - u_{22},\\
F_3(X,U) &= x_3 + u_2 + u_{11} + u_{21}.
\end{aligned}$$

The set $\mathrm{PS}(F_1,F_2,F_3)$ contains $L = 13$ differential polynomials and $\gamma = 0$. The leading matrix S of $\mathcal{P}(X,U)$ has rank 2 and equals

$$S = \begin{bmatrix} -3 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

We consider the perturbation $\phi = (\phi_1(U), \phi_2(U), \phi_3(U))$ with

$$\phi_i(U) = \begin{cases} u_{12} + u_2, & i = 1,\\ u_1, & i = 2,\\ u_{21}, & i = 3. \end{cases}$$

There exists a differential polynomial $P(X) \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$ with coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$ such that the determinant of the 13×13 matrix $M_\phi(13)$ equals

$$\begin{aligned}
&\partial\mathrm{CRes}(F_1^\phi, F_2^\phi, F_3^\phi) = \partial\mathrm{CRes}^h(H_1^\phi, H_2^\phi, H_3^\phi)P(X) =\\
&p\,(1 + 4\,p + 4\,p^4 - p^5 + 2\,p^7 + 11\,p^3 + p^9 - 12\,p^2 - 4\,p^6 + p^8)P(X).
\end{aligned}$$

Then $D_\phi = 1$ and the coefficient of p in $\partial\mathrm{CRes}(F_1^\phi, F_2^\phi, F_3^\phi)$ is

$$A_{D_\phi} = x_{12} - x_2 - 2\,x_{21} - 2\,x_{22} + x_{33} + x_{32} + x_{31} + x_3.$$

We have $A_{D_\phi} = \mathcal{L}_1(x_1) + \mathcal{L}_2(x_2) + \mathcal{L}_3(x_3)$ with

$$\begin{aligned}
\mathcal{L}_1 &= \partial^2 + \partial^3 = \partial^2\,(1+\partial),\\
\mathcal{L}_2 &= -1 - 3\,\partial - 4\,\partial^2 - 2\,\partial^3 = -(\partial+1)\,(2\,\partial^2 + 2\,\partial + 1),\\
\mathcal{L}_3 &= 1 + 2\,\partial + 2\,\partial^2 + 2\,\partial^3 + \partial^4 = (\partial^2+1)\,(\partial+1)^2.
\end{aligned}$$

Therefore $\mathcal{L} = \mathrm{gcld}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) = 1 + \partial$ and $A_\phi = x_{12} - x_2 - 2\,x_{21} - 2\,x_{22} + x_{33} + x_{32} + x_{31} + x_3$ with $\mathrm{c}(A_\phi) = 1$. Then $D_\phi = \mathrm{c}(A_\phi)$. We conclude that the dimension of ID is $n - 1 = 2$ and its implicit equation $A_\phi(X) = 0$.

If we consider the perturbation $\psi = (\psi_1(U), \psi_2(U), \psi_3(U))$ with

$$\psi_i(U) = \begin{cases} u_{22} + u_1, & i = 1,\\ u_2, & i = 2,\\ u_{11}, & i = 3. \end{cases}$$

There exists a differential polynomial $P(X) \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$ with coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$ such that the determinant of the 13×13 matrix $M_\psi(13)$ equals

$$\begin{aligned}
&\partial\mathrm{CRes}(F_1^\psi, F_2^\psi, F_3^\psi) = \partial\mathrm{CRes}^h(H_1^\psi, H_2^\psi, H_3^\psi)P(X) =\\
&-p^2\,(1+p)\,(p^7 + p^6 - 17\,p^5 - 53\,p^4 - 60\,p^3 - 33\,p^2 - 9\,p - 1)P(X).
\end{aligned}$$

Then $D_\psi = 2$ but the coefficient of p^2 is $A_{D_\psi} = A_{D_\phi}$. Therefore $A_\psi = A_\phi$ as expected.

7.2 Example 2

Let $\mathbb{K} = \mathbb{Q}$, $\partial = \frac{\partial}{\partial t}$ and consider the system $\mathcal{P}(X, U)$ of linear DPPEs providing the set of differential polynomials in $\mathbb{K}\{x_1, x_2, x_3, x_4\}\{u_1, u_2, u_3\}$,

$$\begin{aligned}
F_1(X,U) &= x_1 - 2u_1 + u_3 - 3u_{21} + u_{31},\\
F_2(X,U) &= x_2 + 2u_1 - u_3 - u_{11} + 3u_{22} - u_{32},\\
F_3(X,U) &= x_3 + 2u_1 - u_3 + 2u_{21} + u_{32},\\
F_4(X,U) &= x_4 + 2u_1 - u_3 + 3u_{21} - 2u_{31}.
\end{aligned}$$

The set $\mathrm{PS}(F_1, F_2, F_3, F_4)$ contains $L = 18$ differential polynomials and $\gamma = \gamma_1 = 1$. The leading matrix S of $\mathcal{P}(X, U)$ has rank 3 and equals

$$S = \begin{bmatrix} 1 & -3 & -2\\ -1 & 3 & -1\\ 1 & 0 & 0\\ -2 & 3 & 2 \end{bmatrix}.$$

We consider the perturbation $\phi = (\phi_1(U), \phi_2(U), \phi_3(U))$ with

$$\phi_i(U) = \begin{cases} u_{21} + u_3, & i = 1, \\ u_{11} + u_2, & i = 2, \\ u_1, & i = 3, \\ u_{31}, & i = 4. \end{cases}$$

There exists a differential polynomial $P(X) \in (\mathrm{PS}) \cap \mathbb{K}\{X\}$ with coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$ such that the determinant of the 18×18 matrix $M_\phi(18)$ equals

$$\begin{aligned} &\partial\mathrm{CRes}(F_1^\phi, F_2^\phi, F_3^\phi, F_4^\phi) = \partial\mathrm{CRes}^h(H_1^\phi, H_2^\phi, H_3^\phi, H_4^\phi)P(X) = \\ &p^3\,(-2p^9 - 3944p^4 + 789p^2 - 481p^7 - 379p^6 + 108 + 4484p^5 + 212p^8 + p^{11} \\ &- 642p + 527p^3 - 7p^{10})P(X). \end{aligned}$$

Then $D_\phi = 3$ and the coefficient of p^3 in $\partial\mathrm{CRes}(F_1^\phi, F_2^\phi, F_3^\phi, F_4^\phi)$ is

$$\begin{aligned} &A_{D_\phi} = \\ &972x_{12} - 864x_{13} - 972x_{14} - 216x_{22} + 648x_{32} - 972x_{33} + 540x_{42} - 972x_{44}. \end{aligned}$$

We have $A_D = \mathcal{L}_1(x_1) + \mathcal{L}_2(x_2) + \mathcal{L}_3(x_3) + \mathcal{L}_4(x_4)$ with

$$\begin{aligned} \mathcal{L}_1 &= -864\partial^3 + 972\partial^2 - 972\partial^4 = -108\partial^2(8\partial - 9 + 9\partial^2), \\ \mathcal{L}_2 &= -216\partial^2, \\ \mathcal{L}_3 &= -972\partial^3 + 648\partial^2 = -324\partial^2(3\partial - 2), \\ \mathcal{L}_4 &= 540\partial^2 - 972\partial^4 = -108\partial^2(-5 + 9\partial^2). \end{aligned}$$

Therefore $\mathcal{L} = -108\partial^2$ and $A_\phi = 8x_{11} + 9x_{12} - 9x_1 + 2x_2 - 6x_3 + 9x_{31} + 9x_{42} - 5x_4$ with $\mathrm{co}(A_\phi) = 2$. Then $D_\phi > \mathrm{c}(A_\phi)$.

Replace p in $M_\phi(18)$ by zero to obtain $M(18)$ whose principal 18×17 submatrix is M_{L-1}. Compute $L - \mathrm{rank}(M_{L-1}) = 3$. Then $\mathrm{c}(A_\phi) + 1 = L - \mathrm{rank}(M_{L-1})$. We conclude that the dimension of ID is $n - 1 = 3$ and its implicit equation $A_\phi(X) = 0$.

7.3 Example 3

Let $\mathbb{K} = \mathbb{Q}(t)$, $\partial = \frac{\partial}{\partial t}$ and consider the system $\mathcal{P}(X, U)$ of linear DPPEs providing the set of differential polynomials in $\mathbb{K}\{x_1, x_2, x_3\}\{u_1, u_2\}$,

$$\begin{aligned} F_1 &= x_1 - 3 + u_{11} + u_{12} - u_2 - 4\,u_{21} - 3\,u_{22}, \\ F_2 &= x_2 + u_{11} + u_2 - u_{22}, \\ F_3 &= x_3 + 2 + u_{11} + t\,u_2 + u_{21}. \end{aligned}$$

Then the set $\mathrm{PS}(F_1,F_2,F_3)$ contains $L=13$ differential polynomials and $\gamma=0$. The leading matrix S of $\mathcal{P}(X,U)$ has rank 2 and equals

$$S=\begin{bmatrix} -3 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

We consider the perturbation $\phi=(\phi_1(U),\phi_2(U),\phi_3(U))$ with

$$\phi_i(U)=\begin{cases} u_{12}+u_2, & i=1, \\ u_1, & i=2, \\ u_{21}, & i=3. \end{cases}$$

There exists a differential polynomial $P(X)\in(\mathrm{PS})\cap\mathbb{K}\{X\}$ with coefficients in $\mathbb{K}[p]$ and content in $\mathbb{K}$ such that the determinant of the 13×13 matrix $M_\phi(13)$ equals $\partial\mathrm{CRes}(F_1^\phi,F_2^\phi,F_3^\phi)=p\,P(X)$. In this case $\alpha_3\,p=\det(S_3^\phi)\partial\mathrm{CRes}^h(H_1^\phi,H_2^\phi,H_3^\phi)$ where $l(P)=(\alpha_1,\alpha_2,\alpha_3)$ is the leading vector of P. Then $D_\phi=1$ and the coefficient A_{D_ϕ} of p in $\partial\mathrm{CRes}(F_1^\phi,F_2^\phi,F_3^\phi)$ equals

$$A_{D_\phi}=\mathcal{L}_1(x_1-3)+\mathcal{L}_2(x_2)+\mathcal{L}_3(x_3+2)$$

with

$$\begin{aligned}
\mathcal{L}_1 &= -64\,t^2-2656+912\,t-8\,t^3+(468+57\,t^3+33\,t^2-t^5-13\,t^4-522\,t)\,\partial \\
&\quad +(-364+79\,t^2-6\,t-10\,t^3-t^4)\,\partial^2 \\
&\quad +(68\,t+37\,t^2-14\,t^3-t^4-296)\,\partial^3, \\
\mathcal{L}_2 &= (730\,t+860-37\,t^3-229\,t^2+15\,t^4+t^5)\,\partial \\
&\quad +(252\,t+19\,t^4+t^5+1752-421\,t^2+17\,t^3)\,\partial^2 \\
&\quad +(56\,t^3-272\,t-148\,t^2+4\,t^4+1184)\,\partial^3, \\
\mathcal{L}_3 &= -16\,t^2+228\,t-2\,t^3-664+(-2\,t^4+212\,t^2-436\,t-18\,t^3-664)\,\partial \\
&\quad +(-1856-64\,t^3+309\,t^2-5\,t^4+276\,t)\,\partial^2 \\
&\quad +(-2\,t^4+210\,t+32\,t^2-524-32\,t^3)\,\partial^3 \\
&\quad +(-68\,t-37\,t^2+14\,t^3+t^4+296)\,\partial^4.
\end{aligned}$$

Using the Maple package OreTools we check that $(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)=1$, then $A_{D_\phi}=A_\phi$ and $\mathrm{co}(A_\phi)=0$. Replace p in $M_\phi(13)$ to obtain $M(13)$ whose principal 13×12 submatrix is M_{L-1}. Compute $L-\mathrm{rank}(M_{L-1})=2$ which is greater than $\mathrm{c}(A_\phi)+1=1$. We conclude that the dimension of ID smaller than $n-1=2$.

If we take a different perturbation $\psi=(\psi_1(U),\psi_2(U),\psi_3(U))$ with

$$\psi_i(U)=\begin{cases} u_{22}+u_1, & i=1, \\ u_2, & i=2, \\ u_{11}, & i=3. \end{cases}$$

We obtain $D_\psi = 1$ and the coefficient of p in $\partial\mathrm{CRes}(F_1^\psi, F_2^\phi, F_3^\psi)$ is

$$A_{D_\psi} = \mathcal{K}_1(x_1 - 3) + \mathcal{K}_2(x_2) + \mathcal{K}_3(x_3 + 2)$$

with

$$\begin{aligned}
\mathcal{K}_1 = {} & (-76\,t^3 + 154\,t^2 + 12\,t - 156 + 11\,t^4 + t^5)\,\partial \\
& + (204\,t - 100 + t^4 + 8\,t^3 - 94\,t^2)\,\partial^2 \\
& + (-58\,t^2 + 12\,t^3 + 88\,t - 12 + t^4)\,\partial^3, \\
\mathcal{K}_2 = {} & -228\,t + 16\,t^2 + 664 + 2\,t^3 + (-t^5 - 904\,t + 820 + 62\,t^3 + 90\,t^2 - 13\,t^4)\,\partial \\
& + (-t^5 - 17\,t^4 + 346\,t^2 - 892\,t + 412 + 14\,t^3)\,\partial^2 \\
& + (232\,t^2 - 352\,t - 4\,t^4 - 48\,t^3 + 48)\,\partial^3, \\
\mathcal{K}_3 = {} & -228\,t + 16\,t^2 + 664 + 2\,t^3 + (14\,t^3 + 892\,t - 664 + 2\,t^4 - 244\,t^2)\,\partial \\
& + (-156 - 406\,t^2 + 54\,t^3 + 676\,t + 5\,t^4)\,\partial^2 \\
& + (-80\,t^2 + 28\,t^3 + 2\,t^4 + 64 + 60\,t)\,\partial^3 \\
& + (58\,t^2 - 88\,t - t^4 - 12\,t^3 + 12)\,\partial^4.
\end{aligned}$$

Using the Maple package OreTools we check that $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3) = 1$, then $A_{D_\psi} = A_\psi$ and $\mathrm{c}(A_\psi) = 0$. Observe that there is no $\gamma \in \mathbb{K}$ such that $A_\phi = \gamma A_\psi$.

Acknowledgements. The author thanks J. Rafael Sendra for kindly reading and commenting on some parts of this work.

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